Introduction

Example

Find a quadratic polynomial with roots $7-\sqrt{6}$ and $7+\sqrt{6}$.

Fact — If $ax^2+bx+c=0$ has roots α and β, then:

$$\begin{aligned} ax^2+bx+c &= a(x-\alpha)(x-\beta) \\ &= a(x^2-(\alpha+\beta)x+\alpha\beta) \\ &= ax^2-a(\alpha+\beta)x+a\alpha\beta \end{aligned}$$

In particular, $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$

Tip

We don't get anything for free here!

Example

Suppose α and β are roots of $2x^2 + 4x - 5 = 0$, find

(a) $\alpha^2 + \beta^2$

(b) $\frac{1}{\alpha} + \frac{1}{\beta}$

Example

Find a quadratic equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ from the previous example.

Method 1:

Method 2:

Cubic Equations

Fact — If $ax^3 + bx^2 + cx + d = 0$ has roots α, β, γ, then

$$\alpha + \beta + \gamma = -\frac{b}{a}$$
$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$
$$\alpha\beta\gamma = -\frac{d}{a}$$

Tip

These are all **symmetric** functions in α, β, γ, ie if you swap any pair of α, β, γ you still have the same expression

Example

Suppose α, β, γ are roots of $x^3 + 4x + 2 = 0$, find

(a) $\alpha^2 + \beta^2 + \gamma^2$

(b) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

Quartic Equations

Fact — If $ax^4 + bx^3 + cx^2 + dx + e = 0$ has roots $\alpha, \beta, \gamma, \delta$, then

$$\sum \alpha = -\frac{b}{a}$$
$$\sum \alpha\beta = \frac{c}{a}$$
$$\sum \alpha\beta\gamma = -\frac{d}{a}$$
$$\alpha\beta\gamma\delta = \frac{e}{a}$$

Example

Suppose $\alpha, \beta, \gamma, \delta$ are roots of $x^4 + 7x^2 + 2x + 1 = 0$, find a polynomial with roots: $\alpha - 1, \beta - 1, \gamma - 1, \delta - 1$

Example (PPQ)

The cubic equation

$$z^3 + pz^2 + 6z + q = 0$$

has roots α, β and γ.

(a) Write down the value of $\alpha\beta + \beta\gamma + \gamma\alpha$. *(1 mark)*

(b) Given that p and q are real and that $\alpha^2 + \beta^2 + \gamma^2 = -12$:

(i) explain why the cubic equation has two non-real roots and one real root; *(2 marks)*

(ii) find the value of p. *(4 marks)*

(c) One root of the cubic equation is $-1 + 3i$.

Find:

(i) the other two roots; *(3 marks)*

(ii) the value of q. *(2 marks)*

Example (PPQ) (a) Show that $(1+i)^3 = 2i - 2$. *(2 marks)*

(b) The cubic equation

$$z^3 - (5+i)z^2 + (9+4i)z + k(1+i) = 0$$

where k is a real constant, has roots α, β and γ.

It is given that $\alpha = 1+i$.

(i) Find the value of k. *(3 marks)*

(ii) Show that $\beta + \gamma = 4$. *(1 mark)*

(iii) Find the values of β and γ. *(5 marks)*